Solutions to $y'(x) = \cos\left[\pi x^p y(x)^q\right]$

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Abstract

The asymptotic behaviour of solutions to $y'(x) = \cos[\pi x^p y(x)^q]$ is considered. This is a generalisation of the problem of the behaviour of $y'(x) = \cos[\pi x y(x)]$ that was investigated by Bender, Fring and Komijani [1]. We present a derivation of the asymptotic results that follows the approach used in Kerr [2].

1 Introduction

In Bender, Fring and Komijani [1] a detailed asymptotic analysis of the nonlinear initial-value problem

$$y'(x) = \cos[\pi x y(x)], \quad y(0) = a \tag{1}$$

was presented which focused on the solutions for $x \geq 0$. They showed that for $a > 0$ solutions could be split into classes depending on the initial conditions such that solution with $a_{n-1} < a < a_n$ displayed an oscillatory region with n maxima before decaying monotonically to zero. They then found the result that as $n \to \infty$, $a_n \sim 2^{5/6}\sqrt{n}$. This result was subsequently derived using a different approach by Kerr [2]. Here we use this alternative derivation to obtain equivalent results to the generalisation of the original problem where we look for asymptotic solutions to

$$y'(x) = \cos\left[\pi x^p y(x)^q\right], \quad y(0) = a \tag{2}$$

where p and q are positive integers[1]

Much of this derivation is essentially the same as that of Kerr [2]

2 Outline

The typical behaviour of solutions to (2) is essentially the same as the original problem and is shown by the solid lines in figure 1 (with $p = q = 1$). There is an initial oscillatory phase where the frequency increases and the amplitude decreases as the initial value, $y(0)$, increases. These oscillatory solutions drift downwards until they undergo a transition to monotonic decay towards the horizontal axis.

Some of the basic behaviour of the solutions of (2) can be understood by considering the lines in the x–y plane where $x^p y^q$ is constant. The situation is shown schematically

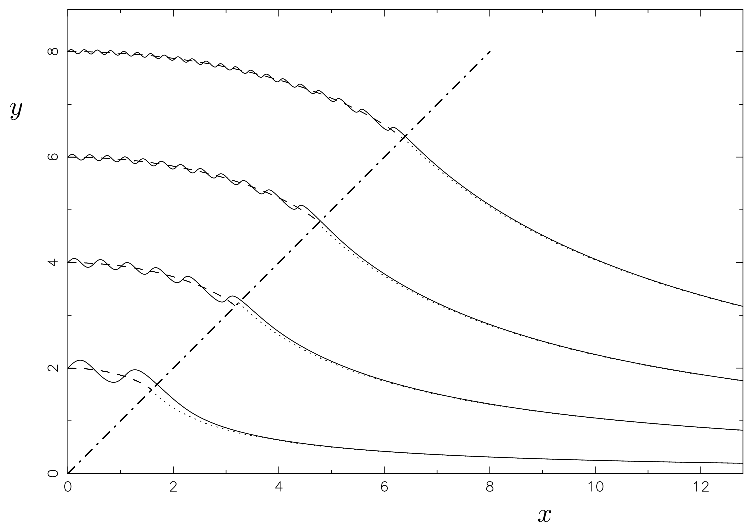

Fig. 1: Plots of solutions to (2) with $y(0) = 2, 4, 6, 8$ and $p = q = 1$. The dotted lines in $x/p > y/q$ show the curves $x^p y^q = C$ to which these converge asymptotically as $x \to \infty$. The dashed lines in $x/p < y/q$ give the estimate of the mean path of the oscillatory part of these curves.

in figure 2. The arguments here are essentially the same as those in section 3 of Kerr, except the region is now divided by the line $x/p = y/q$ and the lines under consideration are lines of the form $x^p y^q = c$.

If we consider lines where $x^p y^q = 2n$ then solutions will have gradient 1 where they intersect these lines, similarly when $x^p y^q = 2n+1$ they will intersect with gradient -1, and when $x^p y^q = 2n \pm 1/2$ they will intersect with gradient 0. The gradients of the solutions will have gradients with magnitude at most 1, while the lines $x^p y^q = c$ for constants c have gradients greater than 1 in magnitude for $x/p < y/q$, and less than 1 for $x/p > y/q$. In the region $x/p < y/q$ solutions must cross the lines $x^p y^q = c$ from left to right, with a maximum each time it crosses a line $x^p y^q = 2n + \frac{1}{2}$, $n = 1, 2, 3, \ldots$. In the region $x/p > y/q$ this restriction no longer holds. This results in the solutions having intrinsically different behaviour above and below the line $x/p = y/q$.

3 Solutions in the region $x/p > y/q$

This is essentially the same argunment as previously in Kerr [2].

Any solution that enters a region $2n + \frac{1}{2} \leq x^p y^q \leq 2n + 1$ is trapped in this region as x increases as the gradient of a solution on the lower boundary is 0, and on the upper boundary is -1. Indeed, in such a region any solution that is initially above a

[1] Strictly speaking this requirement to be positive integers can be relaxed.

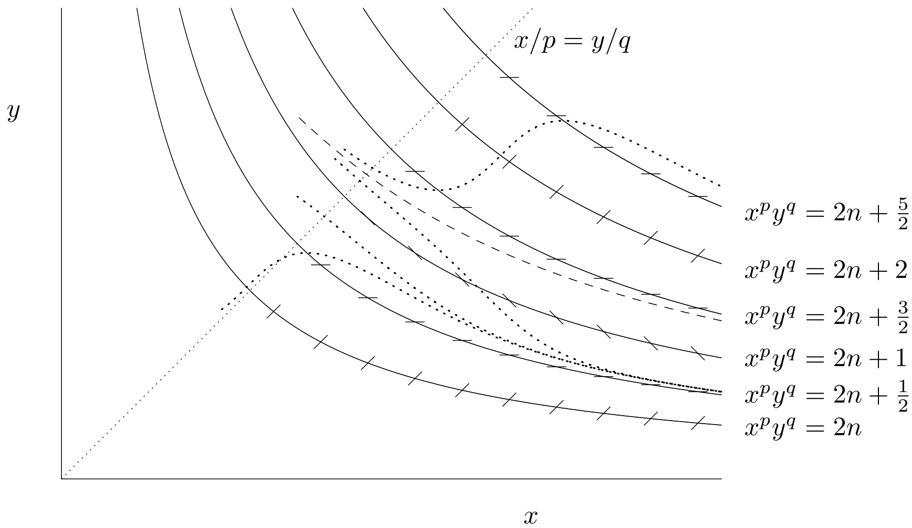

Fig. 2: Schematic plot for solutions in the region $x/p > y/q$ and the influences of the lines of form $x^p y^q = c$. The dotted lines show the trajectories of various solutions, while the dashed line shows the path of the separatrix dividing solutions that converge to $x^p y^q = 2n + 1/2$ from those that converge to $x^p y^q = 2n + 5/2$.

line $x^p y^p = 2n + \frac{1}{2} + \epsilon$ will have a negative gradient of magnitude greater than $\sin \pi\epsilon$ and so must eventually pass below $x^p y^q = 2n + \frac{1}{2} + \epsilon$, whose gradient tends to zero as $x \to \infty$. Thus all solutions in this region asymptote to the line $x^p y^q = 2n + \frac{1}{2}$.

All solutions in the region $2n - \frac{1}{2} < x^p y^q < 2n + \frac{1}{2}$ will have positive gradients and so will pass into the region $2n + \frac{1}{2} \leq x^p y^q \leq 2n + 1$ from below, and will have one maximum in the region $x/p > y/q$.

There is one solution in the region $2n+1 < x^p y^q < 2n + \frac{3}{2}$ that stays in this region. Solutions initially below this curve will pass into the region $2n + \frac{1}{2} \leq x^p y^q \leq 2n+1$ and remain there, while those above it will end up in the region $2n + \frac{5}{2} \leq x^p y^q \leq 2n + 3$. This curve is indicated by the dashed line in figure 2. By a similar argument to that given previously it can be shown that these seperatrices tend towards their asymptotes $x^p y^q = 2n + 3/2$ from below. We will denote the point where the separatrix crosses the line $x/p = y/q$ as $x/p = y/q = b_n$, and hence $2n + 1 < p^p q^q b_n^{p+q} < 2n + 3/2$, or $(2n+1)^{1/(p+q)}/(p^p q^q)^{1/(p+q)} < b_n < (2n + 3/2)^{1/(p+q)}/(p^p q^q)^{1/(p+q)}$.

Clearly, any solution that ends up just above the curve $x^p y^q = 2n + \frac{1}{2}$ will have crossed n lines given by $x^p y^q = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots$ and so will have n maxima. Hence any solution that crosses the line $x/p = y/q$ with $b_{n-1} < x/p = y/q < b_n$ will have n maxima.

4 Solutions in the region $x/p < y/q$

As before, for large values of $y(0)$ the solution $y(x)$ will tend to oscillate quickly with small amplitude. The previous arguments hold here.

If lines of constant $x^p y^q = c$ are given locally by the lines $x + \alpha y = C$ then the mean path of the oscillatory solution is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{1-\alpha^2}-1}{\alpha}. \tag{3}$$

The value of α is determined by the curves $x^p y^q = c$. On such curves

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{pc^{1/q}}{qx^{(p+q)/q}} = -\frac{py}{qx}. \tag{4}$$

Since the gradient of the lines $x + \alpha y = C$ is $-1/\alpha$, we find $\alpha = qx/py$ and so the equation for the slope of the average curve is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\left(\frac{py}{qx}\right)^2 - 1} - \frac{py}{qx}. \tag{5}$$

This has solutions

$$\left(\sqrt{p^2y^2 - q^2x^2} + py\right)^p \left((p+q)y - \sqrt{p^2y^2 - q^2x^2}\right)^{p+q} = 2^p p^p q^{p+q} y(0)^{2p+q}. \tag{6}$$

The solution curves meet the line $x/p = y/q$ at the point $x/p = y/q = \beta$ when

$$y(0) = \frac{(p+q)^{\frac{p+q}{2p+q}}}{2^{\frac{p}{2p+q}}}\beta \tag{7}$$

If the a_n are the values of $y(0)$ which correspond to the b_n, and so are solutions with n maxima then we would have here

$$a_n \approx \frac{(p+q)^{\frac{p+q}{2p+q}}}{2^{\frac{p}{2p+q}}} b_n \approx \frac{2^{\frac{1}{p+q}}(p+q)^{\frac{p+q}{2p+q}} n^{\frac{1}{p+q}}}{2^{\frac{p}{2p+q}} p^{\frac{p}{p+q}} q^{\frac{q}{p+q}}}. \tag{8}$$

5 Conclusions

We have used alternative derivation of Kerr [2] to analyse the generalisation the proiblem considered by Bender, Fring and Komijani [1]. With luk I've got the coefficients right.

References

[1] Carl M. Bender, Andreas Fring, and Javad Komijani. Nonlinear eigenvalue problems. Accepted for publication in *J. Phys. A*, 2014.

[2] Oliver S. Kerr. On "nonlinear eigenvalue problem". Submitted for publication in *J. Phys. A*, 2014.